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SEMISIMPLE AUTOMORPHISM GROUPS OF G -STRUCTURES

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1. Statement of the main result

Let M be a compact manifold of dimension n and suppose we are given a G-structure (possibly of higher order) on M. Thus $G \subset \mathrm{GL}(n, \mathbf{R})$ if the G-structure is of first order, and in general $G \subset \mathrm{GL}(n, \mathbf{R})^{(k)}$ for some k, where $\mathrm{GL}(n, \mathbf{R})^{(k)}$ is the group of k-jets at 0 of diffeomorphisms of $\mathbf{R}^n$ fixing 0. We shall suppose G is a real algebraic group (i.e. the $\mathbf{R}$-points of an algebraic $\mathbf{R}$-group). The G-structure is given by a principal G-bundle $P \to M$ which is a reduction of the bundle of $(k-1)$-jets of frames $P^{(k)} \to M$ to G. We let $\mathrm{Aut}(P)$ be the automorphism group of the G-structure, so $\mathrm{Aut}(P) \subset \mathrm{Diffeo}(M)$. In general of course $\mathrm{Aut}(P)$ is infinite dimensional, although it will be a finite dimensional Lie group in a number of important situations. The point of this paper is to examine (irrespective of whether or not $\mathrm{Aut}(P)$ is a Lie group) which semisimple Lie groups can be subgroups of $\mathrm{Aut}(P)$. In other words, what are the obstructions for a smooth action of a semisimple Lie group to preserve a G-structure? If $G \subset O(n, \mathbf{R})$, then the G-structure is essentially a Riemannian metric and, as is well known, $\mathrm{Aut}(P)$ is then compact. Thus any semisimple Lie group admitting an embedding into $\mathrm{Aut}(P)$ must also be compact. If G is of finite type in the sense of E. Cartan (see [8]), then $\mathrm{Aut}(P)$ is a Lie group and in explicit circumstances one can derive bounds on $\dim(\mathrm{Aut}(P))$. Our main result is the following. For a Lie group Q, $L(Q)$ will denote its Lie algebra.

Theorem 1. *Suppose H is a connected simple Lie group with finite center and with $\mathbf{R}$-$\mathrm{rank}(H) \geqslant 2$. Let M be a compact n-manifold and $P \to M$ a G-structure where $G \subset \mathrm{GL}(n, \mathbf{R})^{(k)}$ (for some k) is a real algebraic subgroup. Suppose that*

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H acts smoothly on M with $H \subset \mathrm{Aut}(P)$ and that H preserves a volume density on M. Then there is an embedding of Lie algebras $L(H) \to L(G)$ (and hence $L(H) \to L(G/\mathrm{rad}(G))$ where $\mathrm{rad}(G)$ is the radical of G).

Thus, roughly speaking, a higher **R**-rank simple Lie automorphism group of P can only be as large as a simple component of G. We remark that the condition that H preserves a volume density will of course follow from the condition $H \subset \mathrm{Aut}(P)$ if the first order part of G is contained in $\mathrm{SL}'(n, \mathbf{R})$ (matrices with $\det^2 = 1$), which is true in many important situations. We also recall that $\mathrm{GL}(n, \mathbf{R})^{(k)} = \mathrm{GL}(n, \mathbf{R}) \ltimes N_{n,k}$ where $N_{n,k}$ is a connected unipotent group. Hence any semisimple subgroup of $G \subset \mathrm{GL}(n, \mathbf{R})^{(k)}$ must actually be a subgroup of $G \cap \mathrm{GL}(n, \mathbf{R})$, i.e. of the first order part of G. In fact, Theorem 1 remains true if we replace $\mathrm{Aut}(P)$ by the group of "automorphisms up to order 1" of P [6].

We do not know whether or not Theorem 1 is true if $\mathbf{R}\text{-rank}(H) = 1$. It seems unlikely that our argument can be made to apply to this case. If one makes the further (strong) assumption that H acts transitively on M then for first order G-structures Theorem 1 can be readily deduced from the Borel density theorem [2], and this argument applies in the **R**-rank 1 case as well. (The proof of Theorem 1 in general in fact makes use of the Borel density theorem, both in the proof of Lemma 6 below and in the proof of Theorem 2, which is quoted from earlier work [11], [13].)

2. Proof of Theorem 1

The proof of Theorem 1 will rely heavily on results from ergodic theory, i.e. on the measure theoretic behavior of the actions in question. We thus begin by recalling two basic results in this direction. The first is a consequence of super-rigidity for measurable cocycles of semisimple Lie group actions which we proved in [11], a generalization of Margulis' super-rigidity theorem [9] (see also [12]). The second is a consequence of the subadditive ergodic theorem, namely, the existence of an "exponent" for a suitable cocycle of a measure preserving transformation of a measure space.

Suppose $P \to M$ is a principal G-bundle, so that G acts on the right of P and $M \cong P/G$. By choosing a measurable section ϕ of P, we obtain a measurable trivialization of P, namely F: $M \times G \to P$ given by $F(m, g) = \phi(m)g$. The action of $\mathrm{Aut}(P)$ on P can then be described as the action on $M \times G$ given by $h(m, g) = (hm, \alpha(h, m)g)$ for $(m, g) \in M \times G$, $h \in \mathrm{Aut}(P)$, and $\alpha(h, m) \in G$. Further, α must satisfy the cocycle identity

$$\alpha(h_1 h_2, m) = \alpha(h_1, h_2 m)\alpha(h_2, m).$$

Two cocycles α, β: $H \times M \to G$ are called equivalent if there is a measurable function f: $M \to G$ such that for all $h \in H$,

$$\alpha(h, m) = f(m)\beta(h, m)f(hm)^{-1}$$

for almost all $m \in M$. The notions of cocycle and equivalence of cocycles clearly make sense for any measurable action of a group on a measure space. The super-rigidity theorem for cocycles [11], [12] implies via the arguments of [12, Chapter 9] or [13, Theorem 2.8] the following.

Theorem 2. *Suppose H is a connected simple Lie group with finite center and $\mathbf{R}$-rank$(H) \geqslant 2$, and S is a (standard) Borel H-space with a finite H-invariant measure. Let $G \subset GL(n, \mathbf{R})$ be a real algebraic group and suppose α: $H \times S \to G$ is a (measurable) cocycle. Then either*:

(a) *considered as a cocycle α: $H \times S \to GL(n, \mathbf{R})$, α is equivalent to a cocycle that takes values in $O(n, \mathbf{R})$; or*

(b) *there is an embedding of Lie algebras $L(H) \to L(G)$.*

The other main ergodic theoretic result we shall need is the existence of the Lyapunov exponent of a tempered cocycle of an action of $\mathbf{Z}$ (see [3], [5], [7] for proofs). Thus, let $\mathbf{Z}$ act on a (standard) probability space (S, μ) and suppose α: $\mathbf{Z} \times S \to GL(n, \mathbf{R})$ is a cocycle. We call α tempered if for each $n \in \mathbf{Z}$ (equivalently for $n = 1$), $s \to \|\alpha(n, s)\|$ and $s \to \|\alpha(n, s)^{-1}\|$ are in $L^\infty(S)$. We then have the following result.

Theorem 3 [5], [7]. *If α: $\mathbf{Z} \times S \to GL(n, \mathbf{R})$ is a tempered cocycle, then*

$$e_\alpha(s) = \lim_{n \to \infty} \frac{1}{n} \log^+ \|\alpha(n, s)\|$$

exists for almost all s. (Here $\log^+ = \max\{0, \log\}$.) We call e_α the exponent of α.

We now make some comments on this result that we will need.

Lemma 4. *Suppose $\alpha \sim \beta$ where α and β are tempered. Then $e_\alpha = e_\beta$ (a.e.).*

Proof. Let $\alpha(n, s) = f(s)\beta(n, s)f(ns)^{-1}$. Choose $A \subset S$ to be of positive measure so that for some $B \in \mathbf{R}$, $\|f(s)\|$, $\|f(s)^{-1}\| \leqslant B$ for $s \in A$. Let $N(A, s) = \{n \in \mathbf{Z}^+ \mid ns \in A\}$. Since there is an invariant probability measure, Poincaré recurrence implies $N(A, s)$ is infinite for almost all $s \in A$. Thus, for almost all $s \in A$ we have

$$\begin{aligned} e_\alpha(s) &= \lim_{\substack{n \to \infty \\ n \in N(A, s)}} \frac{1}{n} \log^+ \left\|f(s)\beta(n, s)f(ns)^{-1}\right\| \\ &\leqslant \varlimsup_{\substack{n \to \infty \\ n \in N(A, s)}} \frac{1}{n}\left(2\log^+ B + \log^+ \|\beta(n, s)\|\right) \\ &\leqslant \varlimsup_{\substack{n \to \infty \\ n \in N(A, s)}} \frac{1}{n} \log^+ \|\beta(n, s)\| = e_\beta(s). \end{aligned}$$

The reverse inequality follows in a similar fashion and hence $e_\alpha(s) = e_\beta(s)$ for a.e. $s \in A$. Letting $B \to \infty$, we can choose $\mu(S)$ arbitrarily close to 1, and hence $e_\alpha = e_\beta$ a.e.

Suppose now that M is a compact n-manifold, and $\phi\colon M \to M$ is a diffeomorphism. Let ξ be a measurable Riemannian metric on M, i.e. for each $m \in M$ an inner product ξ_m on the tangent space TM_m such that ξ_m varies measurably in $m \in M$. If we measurably trivialize TM by measurably choosing an orthonormal basis of TM_m with respect to ξ_m, the action of $\mathbf{Z}$ on TM defined by the powers of ϕ and $d\phi$ yields a cocycle. $\alpha\colon \mathbf{Z} \times \mathrm{M} \to \mathrm{GL}(n, \mathbf{R})$. We will say that α is associated to the measurable metric ξ. The equivalence class of this cocycle is independent of the choice of ξ and the choice of orthonormal basis. If ξ is a continuous metric it is clear that α will be tempered owing to compactness of M. For inner products η, σ on a finite dimensional real vector space we let $M(\eta/\sigma) = \max_{\|X\|_\sigma = 1}\{\|X\|_\eta\}$. Thus if η, σ are measurable Riemannian metrics on M, $M(\eta/\sigma)(s) = M(\eta_s/\sigma_s)$ is a measurable function on M. If η is a measurable Riemnannian metric, we will call η bounded if $M(\eta/\xi)$ and $M(\xi/\eta)$ are in $L^\infty(M)$ for some smooth metric ξ. It is clear that this is independent of the smooth metric involved. The following is straightforward.

Lemma 5. *Suppose η is a bounded measurable Riemannian metric on a compact manifold M, and α is an associated cocycle for some diffeomorphism of M. Then α is tempered.*

To prove Theorem 1 it suffices to consider the case of a first order G-structure (cf. the remarks in the next to last paragraph of §1). Thus, we need only eliminate the possibility of conclusion (a) in Theorem 2 where α: $H \times M \to \mathrm{GL}(n, \mathbf{R})$ is the cocycle defined by the derivative via a measurable trivialization of TM. For $h \in H$, set $\alpha_h = \alpha \mid \{h^n\} \times M$ so we may view α_h as a cocycle $\mathbf{Z} \times M \to \mathrm{GL}(n, \mathbf{R})$. Since $O(n, \mathbf{R})$ is compact, clearly any cocycle taking values in $O(n, \mathbf{R})$ is tempered and has 0 exponent, and hence condition (a) of Theorem 2 would imply via Lemmas 4 and 5 that for all $h \in H$, $e_{\alpha_h} = 0$ a.e. for any cocycle α associated to any bounded measurable Riemannian metric on M. Thus, the proof will follow once we establish the existence of a bounded measurable Riemannian metric with associated cocycle α such that for some $h \in H$, $e_{\alpha_h} \neq 0$.

We will need the following result which has been observed independently by C. C. Moore (private communciation).

Lemma 6. *Let H be a connected simple noncompact Lie group and S a (standard) H-space with finite invariant measure. For $s \in S$, let H_s be the stabilizer of s in H. Then there is a conull set $S_0 \subset S$ such that for $s \in S_0$ either $H_s = H$ or H_s is discrete.*

Proof. Via the ergodic decomposition of the action it suffices to see that the assertion is true in each ergodic component. Thus, we may as well assume that H acts ergodically on S. Let $\mathbb{S}$ be the set of closed subgroups of H. Then $\mathbb{S}$ has the structure of a standard Borel space and the map $s \to H_s$ is a Borel map $S \to \mathbb{S}$ [10]. Let $V = \bigcup_{k=0}^{n} \mathrm{Gr}_k(L(H))$ be the union of the Grassmann varieties of k-planes in the Lie algebra $L(H)$. Then the map $\mathbb{S} \to V$ given by $Q \to L(Q)$ is also Borel. Thus $\phi(s) = L(H_s)$ is a Borel map, and this is clearly an H-map where H acts on $L(H)$ (and hence on V) by the adjoint representation. As observed in [1], the action of H on V is smooth in the sense of ergodic theory, i.e., the orbit space V/H is a countably separated Borel space. (See [4], [12] for a discussion of this notion). (This follows immediately from the fact that $\mathrm{Ad}(H)$ is a subgroup of finite index in a real algebraic group and that the action of $\mathrm{Ad}(H)$ on V is algebraic; thus the orbits are locally closed and this implies that the action is smooth.) We let $\bar{\phi}$ be the composition of ϕ with the natural projection $V \to V/H$. Thus $\bar{\phi}$: $S \to V/H$ is a measurable H-invariant map. Since H acts ergodically on S and V/H is a countably separated Borel space, $\bar{\phi}$ is essentially constant. In other words, by passing to a conull H-invariant subset of S, we can assume $\phi(S)$ lies in a single H-orbit in V. This orbit can be identified with H/H_1, where H_1 is the stabilizer of a point in this orbit, and we thus have a measurable H-map ϕ: $S \to H/H_1$. If the orbit $H/H_1 \subset V$ actually lies in $\mathrm{Gr}_k(L(H))$ for $k = 0$ or $k = \dim L(H)$, then $L(H_s) = 0$ or $L(H_s) = L(H)$ for almost every s, as asserted. Suppose on the other hand that $H/H_1 \subset \mathrm{Gr}_k(L(H))$ for $0 < k < \dim L(H)$. Since H is simple, we must have $H_1 \neq H$. Furthermore $\mathrm{Ad}_H(H_1)$, being the stabilizer in $\mathrm{Ad}_H(H)$ of a k-plane in $L(H)$, is of finite index in a real algebraic group. Thus $\mathrm{Ad}_H(H_1)$, and hence H_1 (since H has a finite center), has only finitely many connected components. However, if μ is a finite H-invariant measure on S, then $\phi_* M$ is a finite H-invariant measure on H/H_1. By the Borel density theorem [2], [12], either H_1 is discrete or $H_1 = H$. Since $H_1 \neq H$, H_1 is discrete, and having only finitely many components, H_1 is finite. This obviously contradicts the existence of a finite invariant measure on H/H_1. Thus $k = 0$ or $\dim(L(H))$ and the proof of the lemma is complete.

Returning to the proof of Theorem 1, by the preceding lemma we can choose an H-invariant set $M_0 \subset M$ of positive measure such that the stabilizers are discrete. For any $m \in M_0$, let $V_m \subset TM_m$ be the tangent space to the H-orbit through m. Fix an inner product on the Lie algebra $L(H)$. For each $m \in M_0$, the map $H \to M$, $h \to hm$ induces an isomorphism of $L(H)$ with V_m, and hence an inner product ξ_m on V_m. The assignment $m \to \xi_m$ is measurable in the obvious sense. Since any measurable function is bounded on a set of large measure, it is clear that there is a bounded measurable Riemannian metric η on

M and a subset $M_1 \subset M_0$ of positive measure such that for $m \in M_1$, η and ξ are equal when restricted to V_m. Suppose now that $h \in H$ and $m \in M_0$ with $hm \in M_0$ as well. Then dh: $TM_m \to TM_{hm}$, $dh(V_m) = V_{hm}$, and it is easy to see that under the above identifications of V_m and V_{hm} with $L(H)$, the map $dh \mid V_m$ corresponds to $\mathrm{Ad}(h)$. Fix some $h \in H$ such that, letting λ be the maximum absolute value of the eigenvalues of $\mathrm{Ad}(h)$, we have $\lambda > 1$. (Since H is a noncompact simple Lie group, such an element always exists.) Since H acts in a finite volume preserving manner on M and M_1 is of positive measure, Poincaré recurrence implies there is a subset $M_2 \subset M_1$, M_2 conull in M_1, so that for every $x \in M_2$ there is a sequence of distinct positive integers n_i (depending on x) with $h^{n_i}x \in M_2$ for all i. Let α be a (tempered) cocycle corresponding to the bounded Riemannian metric η. Then for $s \in M_2$ we have

$$e_{\alpha_h}(s) = \lim_n \frac{1}{n} \log^+ \|\alpha(h^n, s)\|$$
$$= \lim_i \frac{1}{n_i} \log^+ \|\alpha(h^{n_i}, s)\|.$$

By the choice of M_2 and the construction of η,

$$\|\alpha(h^{n_i}, s)\|_\eta \geqslant \|\mathrm{Ad}(h^{n_i})\|.$$

By the choice of h,

$$\lim_n \|\mathrm{Ad}(h^n)\|^{1/n} = \lambda > 1.$$

It follows that for $s \in M_2$, $e_{\alpha_h}(s) > 0$, and this contradiction completes the proof.

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